Lecture 15: Backtracking

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Problem of the Day

The *single-destination shortest path* problem for a directed graph is to find the shortest path *from* every vertex to a specified vertex v. Give an efficient algorithm to solve the single-destination shortest paths problem.

Problem of the Day

Let G be a weighted *directed* graph with n vertices and m edges, where all edges have positive weight. A directed cycle is a directed path that starts and ends at the same vertex and contains at least one edge. Give an $O(n^3)$ algorithm to find a directed cycle in G of minimum total weight. Partial credit will be given for an $O(n^2m)$ algorithm.

Sudoku

							1	2
				3	5			
			6				7	
7						3		
			4			8		
1								
			1	2				
	8						4	
	5					6		

6	7	3	8	9	4	5	1	2
9	1	2	7	3	5	4	8	6
8	4	5	6	1	2	9	7	3
7	9	8	2	6	1	3	5	4
5	2	6	4	7	3	8	9	1
1	3	4	5	8	9	2	6	7
4	6	9	1	2	8	7	3	5
2	8	7	3	5	6	1	4	9
3	5	1	9	4	7	6	2	8

Solving Sudoku

Solving Sudoku puzzles involves a form of exhaustive search of possible configurations.
However, exploiting constraints to rule out certain possibilities for certain positions enables us to *prune* the search to the point people can solve Sudoku by hand.
Backtracking is the key to implementing exhaustive search programs correctly and efficiently.

Backtracking

Backtracking is a systematic method to iterate through all the possible configurations of a search space. It is a general algorithm/technique which must be customized for each individual application.

In the general case, we will model our solution as a vector $a = (a_1, a_2, ..., a_n)$, where each element a_i is selected from a finite ordered set S_i.

Such a vector might represent an arrangement where a_i contains the ith element of the permutation. Or the vector might represent a given subset S, where a_i is true if and only if the ith element of the universe is in S.

The Idea of Backtracking

At each step in the backtracking algorithm, we start from a given partial solution, say, $a = (a_1, a_2, ..., a_k)$, and try to extend it by adding another element at the end.
After extending it, we must test whether what we have so far is a solution.
If not, we must then check whether the partial solution is still potentially extendible to some complete solution.
If so, recur and continue. If not, we delete the last element from a and try another possibility for that position, if one exists.

Recursive Backtracking

```
Backtrack(a, k)
if a is a solution, print(a)
else {
        $k = k + 1$
        compute $S_k$
        while $S_k \neq \emptyset$ do
                $a_k$ = an element in $S_k$
                $S_k$ = $S_k - a_k$
                Backtrack(a, k)
}
```

Backtracking and DFS

Backtracking is really just depth-first search on an implicit graph of configurations.

Backtracking can easily be used to iterate through all subsets or permutations of a set.

Backtracking ensures correctness by enumerating all possibilities.

For backtracking to be efficient, we must prune the search space.

Implementation

```
bool finished = FALSE; (* found all solutions yet? *)

backtrack(int a[], int k, data input)
{
        int c[MAXCANDIDATES]; (* candidates for next position *)
        int ncandidates; (* next position candidate count *)
        int i; (* counter *)

        if (is_a_solution(a,k,input))
                process_solution(a,k,input);
        else {
                k = k+1;
                construct_candidates(a,k,input,c,&ncandidates);
                for (i=0; i<ncandidates; i++) {
                        a[k] = c[i];
                        backtrack(a,k,input);
                        if (finished) return; (* terminate early *)
                }
        }
}
```

is_a_solution(a,k,input)

This Boolean function tests whether the first k elements of vector a are a complete solution for the given problem. The last argument, `input`, allows us to pass general information into the routine.

`construct_candidates(a,k,input,c,nca`

This routine fills an array c with the complete set of possible candidates for the kth position of a, given the contents of the first $k-1$ positions. The number of candidates returned in this array is denoted by `ncandidates`.

`process_solution(a,k)`

This routine prints, counts, or somehow processes a complete solution once it is constructed.

Backtracking ensures correctness by enumerating all possibilities. It ensures efficiency by never visiting a state more than once.

Because a new candidates array c is allocated with each recursive procedure call, the subsets of not-yet-considered extension candidates at each position will not interfere with each other.

Constructing all Subsets

How many subsets are there of an n-element set?

To construct all 2^n subsets, set up an array/vector of n cells, where the value of a_i is either true or false, signifying whether the ith item is or is not in the subset.

To use the notation of the general backtrack algorithm, $S_k = (true, false)$, and v is a solution whenever $k \geq n$.

What order will this generate the subsets of $\{1, 2, 3\}$?

$$(1) \rightarrow (1,2) \rightarrow (1,2,3)* \rightarrow$$

$$(1,2,-)* \rightarrow (1,-) \rightarrow (1,-,3)* \rightarrow$$

$$(1,-,-)* \rightarrow (1,-) \rightarrow (1) \rightarrow$$

$$(-) \rightarrow (-,2) \rightarrow (-,2,3)* \rightarrow$$

$$(-,2,-)* \rightarrow (-,-) \rightarrow (-,-,3)* \rightarrow$$

$$(-,-,-)* \rightarrow (-,-) \rightarrow (-) \rightarrow ()$$

Constructing All Subsets

We can construct the 2^n subsets of n items by iterating through all possible 2^n length-n vectors of *true* or *false*, letting the ith element denote whether item i is or is not in the subset.

Using the notation of the general backtrack algorithm, $S_k = (true, false)$, and a is a solution whenever $k \geq n$.

```
is_a_solution(int a[], int k, int n)
{
        return (k == n); (* is k == n? *)
}

construct_candidates(int a[], int k, int n, int c[], int *ncandidates)
{
        c[0] = TRUE;
        c[1] = FALSE;
        *ncandidates = 2;
}
```

```
process_solution(int a[], int k)
{
        int i; (* counter *)

        printf("(");
        for (i=1; i<=k; i++)
        if (a[i] == TRUE) printf("
        printf(" )");
}
```

Main Routine: Subsets

Finally, we must instantiate the call to `backtrack` with the right arguments.

```
generate_subsets(int n)
{
        int a[NMAX]; (* solution vector *)

        backtrack(a,0,n);
}
```

Constructing all Permutations

How many permutations are there of an n-element set?
To construct all $n!$ permutations, set up an array/vector of n cells, where the value of a_i is an integer from 1 to n which has not appeared thus far in the vector, corresponding to the ith element of the permutation.
To use the notation of the general backtrack algorithm, $S_k = (1, \ldots, n) - v$, and v is a solution whenever $k \geq n$.

$$\begin{aligned}
(1) &\rightarrow (1,2) \rightarrow (1,2,3)* \rightarrow (1,2) \rightarrow (1) \rightarrow (1,3) \rightarrow \\
(1,3,2)* &\rightarrow (1,3) \rightarrow (1) \rightarrow () \rightarrow (2) \rightarrow (2,1) \rightarrow \\
(2,1,3)* &\rightarrow (2,1) \rightarrow (2) \rightarrow (2,3) \rightarrow (2,3,1)* \rightarrow (2,3) \rightarrow () \\
(2) &\rightarrow () \rightarrow (3) \rightarrow (3,1)(3,1,2)* \rightarrow (3,1) \rightarrow (3) \rightarrow \\
(3,2) &\rightarrow (3,2,1)* \rightarrow (3,2) \rightarrow (3) \rightarrow ()
\end{aligned}$$

Constructing All Permutations

To avoid repeating permutation elements, $S_k = \{1, \ldots, n\} - a$, and a is a solution whenever $k = n$:

```
construct_candidates(int a[], int k, int n, int c[], int *ncandidates) {
        int i; (* counter *)
        bool in_perm[NMAX]; (* who is in the permutation? *)

        for (i=1; i<NMAX; i++) in_perm[i] = FALSE;
        for (i=0; i<k; i++) in_perm[ a[i] ] = TRUE;

        *ncandidates = 0;
        for (i=1; i<=n; i++)
                if (in_perm[i] == FALSE) {
                c[ *ncandidates] = i;
                *ncandidates = *ncandidates + 1;
        }
}
```

Auxilliary Routines

Completing the job of generating permutations requires specifying `process_solution` and `is_a_solution`, as well as setting the appropriate arguments to `backtrack`. All are essentially the same as for subsets:

```
process_solution(int a[], int k)
{
        int i; (* counter *)

        for (i=1; i<=k; i++) printf(”
        printf(””);
}

is_a_solution(int a[], int k, int n)
{
        return (k == n);
}
```

Main Program: Permutations

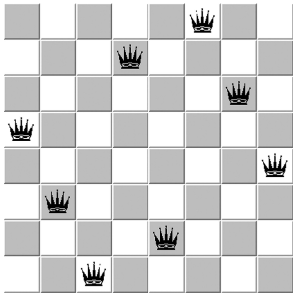
```
generate_permutations(int n)
{
        int a[NMAX]; (* solution vector *)

        backtrack(a,0,n);
}
```

Problem of the Day

A *derangement* is a permutation p of $\{1, \ldots, n\}$ such that no item is in its proper position, i.e. $p_i \neq i$ for all $1 \leq i \leq n$. Write an efficient backtracking program with pruning that constructs all the derangements of n items.

The Eight-Queens Problem



The eight queens problem is a classical puzzle of positioning eight queens on an 8×8 chessboard such that no two queens threaten each other.

Eight Queens: Representation

What is concise, efficient representation for an n-queens solution, and how big must it be?

Since no two queens can occupy the same column, we know that the n columns of a complete solution must form a permutation of n. By avoiding repetitive elements, we reduce our search space to just $8! =$ 40,320 – clearly short work for any reasonably fast machine.

The critical routine is the candidate constructor. We repeatedly check whether the kth square on the given row is threatened by any previously positioned queen. If so, we move on, but if not we include it as a possible candidate:

Candidate Constructor: Eight Queens

```
construct_candidates(int a[], int k, int n, int c[], int *ncandidates)
{
        int i,j; (* counters *)
        bool legal_move; (* might the move be legal? *)

        *ncandidates = 0;
        for (i=1; i<=n; i++) {
                legal_move = TRUE;
                for (j=1; j<k; j++) {
                        if (abs((k)-j) == abs(i-a[j])) (* diagonal threat *)
                                legal_move = FALSE;
                        if (i == a[j]) (* column threat *)
                                legal_move = FALSE;
                }
                if (legal_move == TRUE) {
                        c[*ncandidates] = i;
                        *ncandidates = *ncandidates + 1;
                }
        }
}
```

The remaining routines are simple, particularly since we are only interested in counting the solutions, not displaying them:

```
process_solution(int a[], int k)
{
        int i; (* counter *)

        solution_count ++;
}

is_a_solution(int a[], int k, int n)
{
        return (k == n);
}
```

Finding the Queens: Main Program

```
nqueens(int n)
{
        int a[NMAX]; (* solution vector *)

        solution_count = 0;
        backtrack(a,0,n);
        printf("n=}
```

This program can find the 365,596 solutions for $n = 14$ in minutes.

Can Eight Pieces Cover a Chess Board?

Consider the 8 main pieces in chess (king, queen, two rooks, two bishops, two knights). Can they be positioned on a chessboard so every square is threatened?

			N	B			R
			N				
R							
	B						
		Q			K		

Combinatorial Search

Only 63 square are threatened in this configuration. Since 1849, no one had been able to find an arrangement with bishops on different colors to cover all squares.

We can resolve this question by searching through all possible board configurations *if* we spend enough time.

We will use it as an example of how to attack a combinatorial search problem.

With clever use of backtracking and pruning techniques, surprisingly large problems can be solved by exhaustive search.

How Many Chess Configurations Must be Tested?

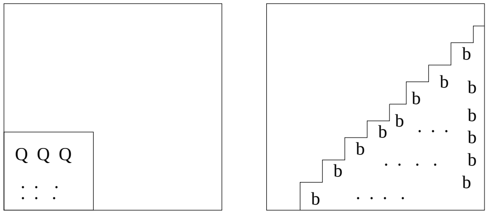
Picking a square for each piece gives us the bound:

$$64!/(64-8)! = 178,462,987,637,760 \approx 10^{15}$$

Anything much larger than 10^8 is unreasonable to search on a modest computer in a modest amount of time.

Exploiting Symmetry

However, we can exploit symmetry to save work. With reflections along horizontal, vertical, and diagonal axis, the queen can go in only 10 non-equivallent positions.
Even better, we can restrict the white bishop to 16 spots and the queen to 16, while being certain that we get all distinct configurations.



$$16 \times 16 \times 32 \times 64 \times 2080 \times 2080 = 2,268,279,603,200 \approx 10^{12}$$

Covering the Chess Board

In covering the chess board, we prune whenever we find there is a square which we *cannot* cover given the initial configuration!

Specifically, each piece can threaten a certain maximum number of squares (queen 27, king 8, rook 14, etc.) We *prune* whenever the number of unthreated squares exceeds the sum of the maximum remaining coverage.

As implemented by a graduate student project, this backtrack search eliminates 95% of the search space, when the pieces are ordered by decreasing mobility.

With precomputing the list of possible moves, this program could search 1,000 positions per second.

End Game

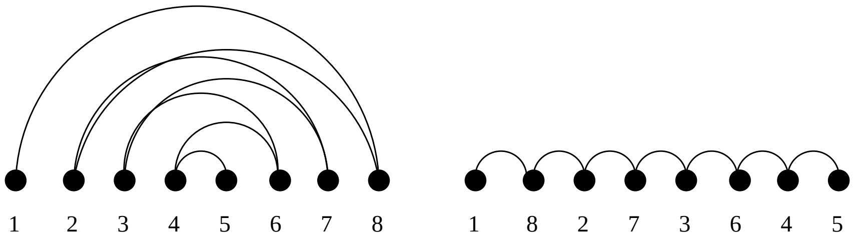
But this is still too slow!

$$10^{12}/10^{3} = 10^{9} \text{ seconds } > 1000 \text{ days}$$

Although we might further speed the program by an order of magnitude, we need to prune more nodes!
By using a more clever algorithm, we eventually were able to prove no solution existed, in less than one day's worth of computing.
You too can fight the combinatorial explosion!

The Backtracking Contest: Bandwidth

The *bandwidth problem* takes as input a graph G, with n vertices and m edges (ie. pairs of vertices). The goal is to find a permutation of the vertices on the line which minimizes the maximum length of any edge.



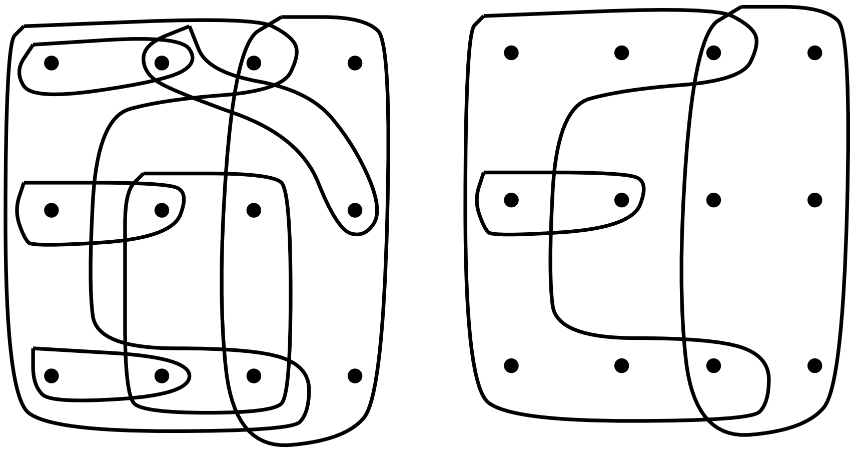
The bandwidth problem has a variety of applications, including circuit layout, linear algebra, and optimizing memory usage in hypertext documents.

The problem is NP-complete, meaning that it is *exceedingly* unlikely that you will be able to find an algorithm with polynomial worst-case running time. It remains NP-complete even for restricted classes of trees.

Since the goal of the problem is to find a permutation, a backtracking program which iterates through all the $n!$ possible permutations and computes the length of the longest edge for each gives an easy $O(n! \cdot m)$ algorithm. But the goal of this assignment is to find as practically good an algorithm as possible.

The Backtracking Contest: Set Cover

The *set cover* problem takes as input a collection of subsets $S = \{S_1, \ldots, S_m\}$ of the universal set $U = \{1, \ldots, n\}$. The goal is to find the smallest subset of the subsets T such that $\cup_{i=1}^{|T|} T_i = U$.



Set cover arises when you try to efficiently acquire or represent items that have been packaged in a fixed set of lots. You want to obtain all the items, while buying as few lots as possible. Finding *a* cover is easy, because you can always buy one of each lot. However, by finding a small set cover you can do the same job for less money.

Since the goal of the problem is to find a subset, a backtracking program which iterates through all the 2^m possible subsets and tests whether it represents a cover gives an easy $O(2^m \cdot nm)$ algorithm. But the goal of this assignment is to find as practically good an algorithm as possible.

Rules of the Game

1. Everyone must do this assignment separately. Just this once, you are not allowed to work with your partner. The idea is to think about the problem from scratch.
2. If you do not completely understand what the problem is, you don't have the *slightest* chance of producing a working program. *Don't be afraid to ask for a clarification or explanation!!!!!*
3. There will be a variety of different data files of different sizes. Test on the smaller files first. Do not be afraid to create your own test files to help debug your program.
4. The data files are available via the course WWW page.

5. You will be graded on how fast and clever your program is, not on style. No credit will be given for incorrect programs.
6. The programs are to run on the whatever computer you have access to, although it must be vanilla enough that I can run the program on something I have access to.
7. You are to turn in a listing of your program, along with a brief description of your algorithm and any interesting optimizations, sample runs, and the time it takes on sample data files. Report the largest test file your program could handle in one minute or less of wall clock time.
8. The top five self-reported times / largest sizes will be collected and tested by me to determine the winner.

Producing Efficient Programs

1. **Don't optimize prematurely:** Worrying about recursion vs. iteration is counter-productive until you have worked out the best way to prune the tree. That is where the money is.

2. **Choose your data structures for a reason:** What operations will you be doing? Is case of insertion/deletion more crucial than fast retrieval?

   When in doubt, keep it simple, stupid (KISS).

3. **Let the profiler determine where to do final tuning:** Your program is probably spending time where you don't expect.